

Parameters of NJL models for a generic representation of the gauge group

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Abstract

We generalize a non-local Nambu-Jona-Lasinio model to a generic representation of the gauge group. The critical temperature is given in a closed form as a function of the parameters of the theory and the cut-off. This result is generally useful in the understanding of QCD-like theories and their thermodynamical behavior.

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QCD is widely accepted as the correct theory describing strong interacting matter. But low energy behavior entails great difficulties to be managed due to its non-perturbative nature. This implies a number of serious problems to understand a full thermodynamical behavior for the vacuum of the theory. So, a different way to overcome such difficulties has been devised in the use of QCD-like theories where an adjoint representation of Fermions is considered as a fundamental one [1–3]. This kind of theories find their most natural place in the technicolor approach to the Standard Model [4].

A fundamental difficulty that arises in this case is that one just formulates a Nambu-Jona-Lasinio (NJL) model but has to postulate the corresponding parameters of the theory relying on plausibility arguments. Quite recently, it has been possible to give a clear proof of the fact that a NJL model is indeed the low-energy limit of QCD and all the parameters are consequently fixed through the parameters of QCD itself [5–7], even if a better understanding of the approximations involved is in need. In [6] we were able to derive a non-local NJL model, founded on a preceding work [8], proving that the instantons liquid model for the vacuum of QCD [9] is an excellent approximation. Then, we obtained an expression for the critical temperature at zero quark mass and chemical potential that perfectly agrees with lattice computations provided the cut-off of the NJL model is taken to be about 770 MeV, a meaningful physical value for this model. The critical temperature obtained in this way agrees very well with a preceding theoretical derivation [10] but we were able to give an explicit value to the mass gap of the theory obtaining a numerical evaluation.

The aim of this paper is to repeat all this for a generic QCD-like model and, in the end, to obtain the value of the cut-off for the critical temperature of about 2 GeV as generally found in literature [11].

Firstly, I point out some notational matter. We use the following table taken from [12]

r	$T(r)$	$C_2(r)$	$d(r)$
F	$\frac{1}{2}$	$\frac{N^2-1}{2N}$	N
A	N	N	$N^2 - 1$
\square	$\frac{N+2}{2}$	$\frac{(N-1)(N+2)}{N}$	$\frac{N(N+1)}{2}$
\square	$\frac{N-2}{2}$	$\frac{(N+1)(N-2)}{N}$	$\frac{N(N-1)}{2}$

TABLE I: Relevant group factors for the representations as taken from [12].

given the generators T_r^a , $a = 1 \dots N^2 - 1$ of the gauge group in the representation r , normalized as $\text{Tr} [T_r^a T_r^b] = T(r) \delta^{ab}$. The quadratic Casimir $C_2(r)$ is given by $T_r^a T_r^a = C_2(r) I$. Here $C_2(r) d(r) = T(r) d(A)$ holds being $d(r)$ the dimension of the representation r .

Then, the results given in [6] can be immediately generalized provided that:

1. *'t Hooft coupling in a given representation is $d(r)g^2$.*
2. *Number of components for the given representation $2NC_2(r)$.*

With these statements the gluon propagator in the Landau gauge is given by[6]:

$$D_{\mu\nu}^{ab}(p) = \delta_{ab} \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \Delta(p^2) + O(1/\sqrt{d(r)}g) \quad (1)$$

being

$$\Delta(p) = \sum_{n=0}^{\infty} \frac{B_n}{p^2 - m_n^2 + i\epsilon} \quad (2)$$

and

$$B_n = (2n+1) \frac{\pi^2}{K^2(i)} \frac{(-1)^{n+1} e^{-(n+\frac{1}{2})\pi}}{1 + e^{-(2n+1)\pi}}. \quad (3)$$

where $K(i) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1+\sin^2 \theta}} \approx 1.3111028777$. The formula for the spectrum of the theory is

$$m_n^{(r)} = \left(n + \frac{1}{2} \right) \frac{\pi}{K(i)} \left(\frac{d(r)g^2}{2} \right)^{\frac{1}{4}} \Lambda_r. \quad (4)$$

From the mass spectrum we can identify a string tension that will be useful in the following.

We can define

$$\sqrt{\sigma_r} = \left(\frac{d(r)g^2}{2} \right)^{\frac{1}{4}} \Lambda_r. \quad (5)$$

Here Λ_r is an arbitrary parameter arising from the integration of the equations of the theory and depending on the chosen representation of $\text{SU}(N)$.

These results show without any doubt that a change of representation in the gauge group produces a difference in the low-energy behavior of the theory. Then, we can use them to fix the Nambu-Jona-Lasinio constant by noting that the non local form factor is [6]

$$\mathcal{G}_r(p) = -\frac{1}{2}g^2 \sum_{n=0}^{\infty} \frac{B_n}{p^2 - (2n+1)^2(\pi/2K(i))^2\sigma_r + i\epsilon} = \frac{G_r}{2}\mathcal{C}_r(p) \quad (6)$$

being G_r the Nambu-Jona-Lasinio constant in the given representation that in our case is given by $G_r = 2\mathcal{G}_r(0) = (g^2/\sigma_r) \sum_{n=0}^{\infty} \frac{B_n}{(2n+1)^2(\pi/2K(i))^2} \approx 0.7854(g^2/\sigma_r)$, so that $\mathcal{C}_r(0) = 1$,

definitely fixed by QCD. The value of σ_r is given in eq.(5). So, finally we can formulate the NJL model for the adjoint representation as [6]

$$\begin{aligned}
S_{QCD}^{(r)} = & \frac{1}{2} \int d^4x \left[\frac{1}{2}(\partial\sigma)^2 - \frac{1}{2}m_0^2\sigma^2 \right] + \int d^4x \sum_q \bar{q}(x) i \not{D} q(x) \\
& - g^2 \Lambda_r \int d^4x d^4y' \Delta(x-y') \sum_q \bar{q}(x) T_r^a \gamma^\mu q(x) \sum_{q'} \bar{q}'(y') T_r^a \gamma_\mu q'(y') \\
& + O(1/\sqrt{d(r)}g).
\end{aligned} \tag{7}$$

The constant Λ_r can be fixed through the value of the critical temperature for each representation as we will do in a moment. From this it is clear that the following holds

$$\frac{G_A}{G_F} = \frac{\sigma_F}{\sigma_A} = \sqrt{\frac{d(F)}{d(A)} \frac{\Lambda_F}{\Lambda_A}} = \sqrt{\frac{N}{N^2-1} \frac{\Lambda_F}{\Lambda_A}}. \tag{8}$$

The condition $\Lambda_A = \Lambda_F$ should be eventually proved with lattice computations or just guessed. Then, we can immediately put down the equation for the critical temperature for this model coming from the gap equation as[6]

$$\frac{4d(r)N_f}{m_0^{(r)2} + 1/G_r} T_{c-r} \sum_{k=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} \mathcal{C}_r^2(\omega_k, \mathbf{p}) \frac{1}{\omega_k^2 + \mathbf{p}^2} = 1. \tag{9}$$

being $m_0^{(r)} \approx 1.19\sqrt{\sigma_r}$, $G_r \approx 0.7854(g^2/\sigma_r)$ as seen above and the Matsubara frequencies are $\omega_k = (2n+1)\pi T_{c-r}$. The cut-off Λ_{NJL-r} of the integral is the physical parameter to be computed given T_{c-r} . In order to evaluate the integral, we just note that the above equation can be rewritten as (in the Euclidean metric)

$$\begin{aligned}
& \frac{4d(r)N_f}{m_0^{(r)2} + 1/G_r} T_{c-r} \frac{1}{2\pi^2} \frac{g^4}{G_r^2} \sum_{k=-\infty}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} B_{n_1} B_{n_2} \times \\
& \int_0^\Lambda dp p^2 \frac{1}{\omega_k^2 + \mathbf{p}^2} \frac{1}{\omega_k^2 + \mathbf{p}^2 + m_{n_1}^2} \frac{1}{\omega_k^2 + \mathbf{p}^2 + m_{n_2}^2} = 1.
\end{aligned} \tag{10}$$

The integral can be evaluated to give

$$\begin{aligned}
I(\omega_k, n_1, n_2) = & \left((\omega_k^2 + m_{n_2}^2)^{\frac{1}{2}} \arctan \left(\frac{\Lambda}{(\omega_k^2 + m_{n_2}^2)^{\frac{1}{2}}} \right) m_{n_1}^2 (\omega_k^2 + m_{n_1}^2)^{\frac{1}{2}} \right. \\
& + \omega_k \arctan \left(\frac{\Lambda}{\omega_k} \right) (\omega_k^2 + m_{n_1}^2)^{\frac{1}{2}} (m_{n_2}^2 - m_{n_1}^2) \\
& - \arctan \left(\frac{\Lambda}{(\omega_k^2 + m_{n_1}^2)^{\frac{1}{2}}} \right) \omega_k^2 m_{n_2}^2 \\
& \left. - \arctan \left(\frac{\Lambda}{(\omega_k^2 + m_{n_1}^2)^{\frac{1}{2}}} \right) m_{n_1}^2 m_{n_2}^2 \right) \frac{1}{m_{n_2}^2 (m_{n_1}^2 - m_{n_2}^2) m_{n_1}^2 (\omega_k^2 + m_{n_1}^2)^{\frac{1}{2}}}.
\end{aligned} \tag{11}$$

Now, we assume that the critical temperature is much higher than the mass terms and we are able to approximate the above integral as

$$I(\omega_k, n_1, n_2) \approx I(\omega_k, n_1, n_1) \approx \frac{\Lambda^3}{3\omega_k^6} \quad (12)$$

where use has been made of the identity $\sum_{n=0}^{\infty} B_n = 1$. Then, we can give the critical temperature in a closed form as

$$T_{c-r}^5 \approx \frac{1}{720\pi^2} \frac{d(r)N_f}{m_0^{(r)2} + 1/G_r} \frac{g^4}{G_r^2} \Lambda^3 \quad (13)$$

using the fact that $\sum_{n=-\infty}^{\infty} 1/(2n+1)^6 = \pi^6/480$. Our aim is to obtain the cut-off Λ once the critical temperature is given. We assume that, as far as basic parameters of the gluonic part are considered, we can take a conservative step assuming it to be unchanged through different representations. So, $\Lambda_r = \Lambda_F = \Lambda_A = \dots$ and the same for the coupling that we take $g \approx 3$ and $\sigma_F \approx (0.44)^2 \text{ GeV}^2$ for SU(3). Then, $\sigma_A = \sqrt{\frac{N^2-1}{N}}\sigma_F$ and the mass gap moves to higher energies, $m_0^{(A)} \approx 0.67 \text{ GeV}$, while NJL coupling is lowering. With these values, for SU(3) with two flavors and taking $T_{c-A} = 2 \text{ GeV}$ one has immediately $\Lambda_{NJL-A} \approx 34 \text{ GeV}$ for the adjoint representation. It is easy to see that $m_0^{(A)}/\Lambda_{NJL-A} \approx 0.02$.

For aims of completeness, we extend this numerical study to the sextet case (symmetric representation). We will get $\sigma_S = \sqrt{\frac{N+1}{2}}\sigma_F$. This means that $m_0^{(S)} \approx 0.62 \text{ GeV}$. For $g \approx 3$, fixing $T_{c-S} = 0.7 \text{ GeV}$ one gets $\Lambda_{NJL-S} \approx 6.88 \text{ GeV}$. This gives $m_0^{(S)}/\Lambda_{NJL-S} \approx 0.09$.

We have shown how the parameters of a Nambu-Jona-Lasinio model in a generic representation of the gauge group can be obtained from QCD with the parameters properly fixed. This has permitted us to obtain the critical temperature for this case and so, to determine the physical cut-off of the theory. This should help to frame better further analysis in this line of research.

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- [1] T. Zhang, T. Brauner and D. H. Rischke, JHEP **1006**, 064 (2010).
 - [2] H. Nishimura and M. C. Ogilvie, J. Phys. Conf. Ser. **312**, 012011 (2011).
 - [3] H. Nishimura and M. C. Ogilvie, Phys. Rev. D **81**, 014018 (2010).

- [4] J. R. Andersen, O. Antipin, G. Azuelos, L. Del Debbio, E. Del Nobile, S. Di Chiara, T. Hapola and M. Jarvinen *et al.*, Eur. Phys. J. Plus **126**, 81 (2011).
- [5] K. -I. Kondo, Phys. Rev. D **82**, 065024 (2010).
- [6] M. Frasca, Phys. Rev. C **84**, 055208 (2011).
- [7] M. Frasca, Int. J. Mod. Phys. E **18**, 693 (2009).
- [8] T. Hell, S. Roessner, M. Cristoforetti and W. Weise, Phys. Rev. D **79**, 014022 (2009).
- [9] T. Schafer and E. V. Shuryak, Rev. Mod. Phys. **70**, 323 (1998).
- [10] D. Gomez Dumm and N. N. Scoccola, Phys. Rev. C **72**, 014909 (2005).
- [11] F. Karsch and M. Lutgemeier, Nucl. Phys. B **550**, 449 (1999).
- [12] T. A. Ryttov and F. Sannino, Phys. Rev. D **78**, 065001 (2008).